The Cichoń Diagram for Degrees of Relative Constructibility

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Abstract

Following a line of research initiated in [4], I describe a general framework for turning reduction concepts of relative computability into diagrams forming an analogy with the Cichoń diagram for cardinal characteristics of the continuum. I show that working from relatively modest assumptions about a notion of reduction, one can construct a robust version of such a diagram. As an application, I define and investigate the Cichoń Diagram for degrees of constructibility relative to a fixed inner model W. Many analogies hold with the classical theory as well as some surprising differences. Along the way I introduce a new axiom stating, roughly, that the constructibility diagram is as complex as possible[1].

1 Introduction

In [4] an analogue of the Cichoń diagram was developed for highness properties of Turing degrees. In this paper I show that the framework set up in [4] is very flexible and can be used to produce a wide variety of Cichoń Diagrams for various reductions related to various notions of computability. Expanding upon this more general viewpoint I show that such diagrams exist for many of the standard reduction concepts on the reals. In each case I obtain an analogue of (a large fragment of) the Cichoń diagram. As an example, I show how such a diagram can be constructed and studied for degrees of constructibility relative to some inner model W alongside the corresponding reduction $\leq_W$. I also indicate how such diagrams could be constructed in other contexts, though I leave the details to future projects.

[1]This reasearch was supported by a CUNY mathematics fellowship and the author would like to thank the mathematics department at the Graduate Center at CUNY for this. The author would also like to express his gratitude to Professor Joel David Hamkins for his patient, enthusiastic and thoughtful help and encouragement as well as Professors Gunter Fuchs and Alfred Dolich for sitting on the author's oral exam committee where a version of this material was originally presented and Professor Jörg Brendle for very helpful comments on an earlier draft.
MSC2010 Classification: 03E35, 28A05, 03E65

$$\begin{array}{ccccccc}
 & & \mathcal{B}_{\leq_W}(\neq^*) & \longrightarrow & \mathcal{D}_{\leq_W}(\in^*) & \longrightarrow & \omega^\omega \setminus (\omega^\omega)^W \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{B}_{\leq_W}(\leq^*) & \longrightarrow & \mathcal{D}_{\leq_W}(\leq^*) & & \\
 & & \uparrow & & \uparrow & & \\
\emptyset & \longrightarrow & \mathcal{B}_{\leq_W}(\in^*) & \longrightarrow & \mathcal{D}_{\leq_W}(\neq^*) & &
\end{array}$$

Figure 1: The Cichoń diagram for $\leq_W$

Let W be a transitive inner model of ZFC. Recall that for reals x and y (in V) the relation $x \leq_W y$ is defined by $x \in W[y]$. The following theorem, which I formalize in GBC, is the first of two main theorems I prove.

Main Theorem 1. *For any transitive inner model W of ZFC, the inclusions shown in the Cichoń diagram for $\leq_W$ (Figure 1) all hold. Furthermore, the diagram is complete in the sense that there is a forcing extension showing that no other implications are necessarily true.*

In fact I show more: for a wide variety of computability-like notions alongside their corresponding reduction concepts one can construct a Cichoń diagram similar to the one pictured above. In the case of $\leq_W$ the theory and corresponding diagram are robust in that they interact very well with regards to the familiar forcings to add reals that one studies for the classical Cichoń diagram. Moreover by a simple forcing over the inner model W the diagram can be saturated in the sense that all possible separations can be realized simultaneously and this can be done in such a way that is indestructible with respect to further forcing. This is the second main theorem of this paper.

Main Theorem 2. *There is a model of ZFC realizing simultaneously all possible separations between nodes of the Cichoń diagram for $\leq_W$. Moreover, no further forcing over this model can destroy this property.*

I dub the statement that "all possible separations of the $\leq_W$ Cichoń diagram are realized", $CD(\leq_W)$. The paper finishes by briefly treating $CD(\leq_W)$ as an axiom.

2 Generalized Cichoń Diagrams for Reductions

In this section I expand on the general viewpoint of computable reduction concepts as giving rise to Cichoń diagrams. Underlying the construction of such Cichoń diagrams for reduction concepts is a certain perspective on cardinal characteristics of the continuum. To describe this perspective better, let us think of cardinal characteristics of the continuum in terms of small and large sets relative to some relation giving this notion of smallness and largeness. For example, recall that the binary relation $\leq^*$ is defined on ω^ω as $f \leq^* g$ if and only if for all but finitely many $n \in \omega$, $f(n) \leq g(n)$. A family reals A is $(\leq^*)$ -*unbounded* if for all $f \in \omega^\omega$ there is some $g \in A$ such that $g \not\leq^* f$. The smallest cardinality of an unbounded family is called the unbounding number, denoted $\mathfrak{b} = \mathfrak{b}(\leq^*)$. Dually, a family of reals $A \subseteq \omega^\omega$ is $(\leq^*)$ -*dominating* if for all $g \in \omega^\omega$ there is a $f \in A$ such that $g \leq^* f$. The least size of a dominating family is called the dominating number, denoted $\mathfrak{d} = \mathfrak{d}(\leq^*)$. Intuitively one thinks of bounded families as being "small" and dominating families as being "big". Thus, heuristically one might think of $\mathfrak{b}$ as the least size of a set that's not "small" and $\mathfrak{d}$ as the least size of a set that's "big". To obtain an analogy in the computable world, the authors of [4] define $\mathcal{B}(\leq^*)$ as the set of oracles computing a function f such that $g \leq^* f$ for each computable function g and $\mathcal{D}(\leq^*)$ as the set of oracles computing a function f such that $f \not\leq^* g$ for all computable g. In other words $\mathcal{B}(\leq^*)$ is the set of oracles which can compute a witness to the fact that the computable functions are "small" and $\mathcal{D}(\leq^*)$ is the set of oracles which can compute a witness to the fact that the computable functions are not "big". Moreover, these sets turn out to correspond to "highness" properties of Turing degrees that are well studied in computability theory. Specifically, by a theorem of Martin (cf [4, pp. 3]), $\mathcal{B}(\leq^*)$ is the set of high degrees and, by definition, $\mathcal{D}(\leq^*)$ is the set of hyperimmune degrees.

My key observation is that this formalism has nothing to do with *Turing* computability per se. This motivates the following general definition.

Definition 2.1. A *reduction concept* is a triple $(X, \sqsubseteq, 0)$ where X is a nonempty set, $0 \in X$ is some distinguished element and $\sqsubseteq$ is a partial pre-order on X. If X is given or implicit, we also say that the pair $(\sqsubseteq, 0)$ is a *reduction concept on* X. If $(X, \sqsubseteq, 0)$ is a reduction concept, then for $x, y \in X$ say that x is $\sqsubseteq$-*reducible to* y if $x \sqsubseteq y$ and say that x is $\sqsubseteq$-*basic* if it is $\sqsubseteq$-reducible to 0.

Let $(\sqsubseteq, 0)$ be a reduction concept on X and $R \subseteq X \times X$ be a binary relation. Let $\sqsubseteq\restriction 0 = \{y \in X \mid y \sqsubseteq 0\}$ be the basic reals. Then define the *bounding set* for R as

$$\mathcal{B}_\sqsubseteq(R) = \{x \in X \mid \exists y \sqsubseteq x \, \forall z \in \sqsubseteq\restriction 0 \, [zRy]\}$$

and the *non-dominating set* for R as

$$\mathcal{D}_\sqsubseteq(R) = \{x \in X \mid \exists y \sqsubseteq x \, \forall z \in \sqsubseteq\restriction 0 \, [\neg yRz]\}.$$

Roughly, if $\sqsubseteq$ is some sort of relative computability relation, then $\mathcal{B}_\sqsubseteq(R)$ is the set of elements of $x \in X$ which compute an R-bound on the computable elements of X and $\mathcal{D}_\sqsubseteq(R)$ is the set of $x \in X$ which compute an element which is not R-dominated by the set of all computable elements. If R is a relation giving a notion of "small" and "big" sets as described above one can think of $\mathcal{B}_\sqsubseteq(R)$ as the set of elements computing a witness to the fact that the $\sqsubseteq$-basic sets are small and $\mathcal{D}_\sqsubseteq(R)$ as the set of elements computing a witness to the fact that the $\sqsubseteq$-basic elements are not big.

Example 2.2 ([4])**.** Let $0 \in \omega^\omega$ be some computable real, say the constant function at 0. Then the pair $(\leq_T, 0)$ forms a reduction concept on the reals. The basic reals are the computable reals. For any binary relation R on the reals $\mathcal{B}_{\leq_T}(R)$ is the set of Turing degrees computing an element of X which R-bounds all the computable sets. Similarly $\mathcal{D}_{\leq_T}(R)$ is the set of Turing degrees computing an element of X which is not R-dominated by any computable set.

The next example will be the central focus of the rest of this article.

Example 2.3. Let $0 \in \omega^\omega$ be constructible. Then the pair $(\leq_L, 0)$ is a reduction concept on ω^ω where $x \leq_L y$ if $x \in L[y]$. The basic reals are the constructible reals. More generally, fix some inner model $W \subseteq V$ and let $\leq_W$ be constructibility relative to W. Then if $0 \in (\omega^\omega)^W$ is any given real in W the pair $(\leq_W, 0)$ forms a reduction concept on Baire space and the basic reals are those of W. Since this is the main case let me explicit what the bounding and non-dominating sets are. Let R be a relation on the reals of V. The set $\mathcal{B}_{\leq_W}(R)$ consists of all reals x in V such that in $W[x]$ there is an R-bound on the reals of W. Similarly the set $\mathcal{D}_{\leq_W}(R)$ consists of all reals x in V such that in $W[x]$ there is a real which is not R-bounded by any real in W. For example, $\mathcal{B}_{\leq_W}(\leq^*)$ is the set of dominating reals over W in V and $\mathcal{D}_{\leq_W}(\leq^*)$ is the set of unbounded reals over W in V.

I will come back to this example in the next section. First, let me give some more examples of reduction concepts on the reals, though I will not treat them in detail in this article.

Example 2.4. Recall that for $x, y \in \mathcal{P}(\mathbb{N})$, the relation $\leq_A$ is defined by $x \leq_A y$ if and only if x is definable in the standard model of arithmetic with an extra predicate for y. The pair $(\leq_A, \emptyset)$ forms a reduction concept on $\mathcal{P}(\mathbb{N})$. In this case the basic reals are the sets which are $\emptyset$-definable in the standard model of arithmetic. More generally this could be done with any model of PA.

Example 2.5. Recall that the relation of many-one polytime reduction, $\leq_m^p$ is defined by $x \leq_m^p y$ if and only if there is a function f which is computable in polynomial time such that $n \in x$ if and only if $f(n) \in y$. The pair $(\leq_M^p, \emptyset)$ is a reduction concept on $\mathcal{P}(\mathbb{N})$.

Example 2.6. Let $\kappa > \omega$ be an uncountable cardinal. Recently there has been much work in the descriptive set theory of "generalized" Baire and Cantor spaces, κ^κ and 2^κ, including various generalizations of cardinal characteristics of the continuum. The same can be done in my framework for degrees of constructibility. For instance notions of eventual domination, etc all make sense in the general context of κ^κ and corresponding bounding and non-dominating sets can be constructed over the basic elements, $(\kappa^\kappa)^L$.

The framework described above is flexible enough that $(X, \sqsubseteq, 0)$ need not be some actual notion of computability on the reals nor have an explicit relation to cardinal characteristics of the continuum. For instance one might consider a class of models of a fixed theory in a fixed language with embeddibility. In this case, depending on the relations R one studied, one would arrive at a diagram corresponding to when models with certain properties embed into one another. There are many possibilities, each giving a potentially interesting "Cichoń diagram" of inclusions between the various bounding and non-dominating sets for an appropriate collection of relations. In future work I hope to explore all of these more fully.

Presently however, let me restrict my attention to the types of cases described in the preceding examples. To see how these examples can lead to "Cichoń diagrams" let me define some relations.

Definition 2.7 (Combinatorial relations)**.** I consider the reals as elements of Baire space, ω^ω. Let f, g be reals. Then

1. $f \neq^* g$ if there is some k such that for all $l > k$ $f(l) \neq g(l)$. In this case say that g is *eventually not equal to f*. Note that the negation of $\neq^*$ is *infinitely often equal*, not eventual equality.

2. Let $h \in \omega^\omega$ and recall that an *h-slalom* is a function $\sigma : \omega \to [\omega]^{<\omega}$ such that for all $n \in \omega$ the set $|\sigma(n)| \leq h(n)$. In the case where h is the identity function call σ simply a *slalom*. For a slalom σ, I write $f \in^* \sigma$ if there is some k such that for all $l > k$ $f(l) \in \sigma(l)$. In this case say that f *is eventually captured by σ*.

Even in this general framework I can now prove a collection of implications giving a version of the Cichoń diagram.

Theorem 2.8. *Let $(\sqsubseteq, 0)$ be a reduction concept on ω^ω extending $\leq_T$ such that if $f, g \sqsubseteq h$ then $f \circ g \sqsubseteq h$ then, interpreting arrows as inclusions, the following all hold:*

$$\begin{array}{ccccccc}
 & & \mathcal{B}_{\sqsubseteq}(\neq^*) & \longrightarrow & \mathcal{D}_{\sqsubseteq}(\in^*) & \longrightarrow & \omega^\omega \setminus \{x \mid x \sqsubseteq 0\} \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{B}_{\sqsubseteq}(\leq^*) & \longrightarrow & \mathcal{D}_{\sqsubseteq}(\leq^*) & & \\
 & & \uparrow & & \uparrow & & \\
\emptyset & \longrightarrow & \mathcal{B}_{\sqsubseteq}(\in^*) & \longrightarrow & \mathcal{D}_{\sqsubseteq}(\neq^*) & &
\end{array}$$

Figure 2: A Cichoń diagram for an arbitrary reduction concept on Baire space

Proof. Note that slaloms can be computably coded by reals so, since the relation $\sqsubseteq$ extends Turing computability the $\in^*$ can be seen as a relation on the reals. I drop the $\sqsubseteq$ subscript for readability. Also, I'll write "basic" for $\sqsubseteq$-basic and if $y \sqsubseteq x$ then I'll say that "x builds y". The requirement that $\sqsubseteq$ be closed downwards under compositions will be used implicitly throughout the argument where I will show that a function can build two other functions hence it can build their composition.

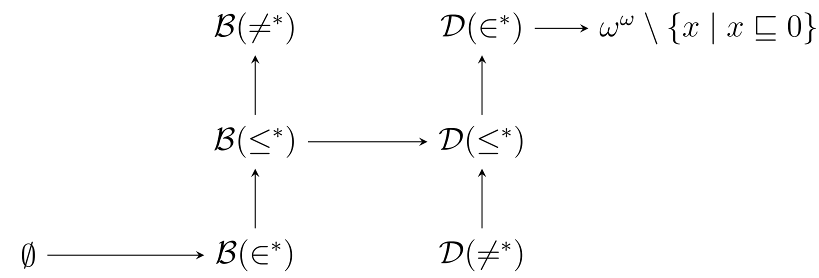


Figure 3: The easy cases

Let's begin with the easy cases, as shown in Figure 3. First I'll show that $\mathcal{B}(\in^*) \subseteq \mathcal{B}(\leq^*)$ or that every x building a slalom eventually capturing all the basic reals builds a real which eventually dominates all basic reals. This is proved as follows. Suppose $x \in \mathcal{B}(\in^*)$ and let $\sigma \sqsubseteq x$ be a slalom witnessing this. Then, define $z(n) = \max(\sigma(n))+1$. Notice that $z \leq_T \sigma$ so $z \sqsubseteq \sigma$ and hence $z \sqsubseteq x$. Moreover, since σ eventually captures all basic reals, z must eventually dominate them all so $x \in \mathcal{B}(\leq^*)$.

Now let's show that $\mathcal{B}(\leq^*) \subseteq \mathcal{B}(\neq^*)$ or that if x can build a real eventually dominating all basic reals then it can build a real eventually different real from all basic reals. Notice however that once stated like this the proof simply the observation that if x eventually dominates y and $y+1$ (which is basic if y is since $\leq_T$ is extended by $\sqsubseteq$), then, in particular x is eventually different from y.

Next let's show $\mathcal{B}(\leq^*) \subseteq \mathcal{D}(\leq^*)$ or that if x builds a real which eventually dominates all basic reals then it builds a real which is not dominated by any basic real. But now stated like this it's obvious.

Next I show that $\mathcal{D}(\neq^*) \subseteq \mathcal{D}(\leq^*)$ or that if there is a real which is equal to every basic real infinitely often then there is a real which is never dominated by any basic real. This is obvious though since if x were dominated by some basic real y then it could not be infinitely-often-equal to the basic real $y+1$.

Now I show that $\mathcal{D}(\leq^*) \subseteq \mathcal{D}(\in^*)$ or that if x builds a real which is not dominated by any basic real then it builds a real that is never eventually captured by any basic slalom. Suppose $x \in \mathcal{D}(\leq^*)$ and let $y \sqsubseteq x$ witness this. Then, if σ is a basic slalom, let z be defined by $z(n)$ is one plus the sum of the elements in $\sigma(n)$. Note that $z \leq_T \sigma$ so z is basic. Thus there are infinitely many n such that $y(n) \geq z(n)$ so y cannot be eventually captured by σ.

The last easy inclusion, that all the reals in every node are not themselves basic is completely straightforward. For instance, if $x \in \mathcal{D}(\in^*)$ is a real which is not eventually captured by any basic slalom then of course x is not basic since if it were, the slalom $n \mapsto \{x(n)\}$ would be as well.

Now I move on to the more difficult inclusions, starting with $\mathcal{B}(\in^*) \subseteq \mathcal{D}(\neq^*)$. Substantively this states that if a real x builds a slalom eventually capturing all basic functions then x also builds a real which is infinitely-often-equal to all basic functions. In fact I will show a more general claim that implies this. The following lemma and proof is essentially a reinterpretation of Theorem 1.5 from [1].

Lemma 2.9. *For any real x the following are equivalent.*

1. *There is a real $g \sqsubseteq x$ such that for all basic $f \in \omega^\omega$, there exist infinitely many $n \in \omega$ such that $g(n) = f(n)$*

2. *There is a basic $h \in \omega^\omega$ and an h-slalom $\sigma \sqsubseteq x$ such that for all basic $f \in \omega^\omega$ there are infinitely many $n \in \omega$ such that $f(n) \in \sigma(n)$.*

Moreover, given an infinitely-often-equal real as in 1)*, one can build from it an h-slalom as in* 2) *and given an h-slalom σ as in* 2) *one can build an infinitely-often-equal real as in* 1)*. Thus, $x \in \mathcal{D}(\neq^*)$ if and only if there is a basic $h \in \omega^\omega$ and an h-slalom which captures each of the basic reals infinitely often.*

Before proving Lemma 2.9, notice that it implies the inclusion $\mathcal{B}(\in^*) \subseteq \mathcal{D}(\neq^*)$ since any slalom which captures every basic real cofinitely often must in particular capture each basic real infinitely often so if $x \in \mathcal{B}(\in^*)$ builds such a slalom, by the lemma x must be able to build an infinitely-often-equal real as well.

Proof of Lemma 2.9. The forward direction is obvious: suppose that g is an infinitely-often-equal real. Then clearly the 1-slalom $\phi : \omega \to [\omega]^1$ such that $\phi(n) = \{g(n)\}$ is $\leq_T$-computable from g and hence $\sqsubseteq$-reducible to g, thus giving the desired h-slalom.

For the backward direction fix a basic real h such that there exists an h-slalom as in the statement of 2. I need to find a real g which is infinitely often equal to every basic real. In a basic fashion, fix a family of finite, nonempty, pairwise disjoint subsets of ω enumerated $\{J_{n,k} \mid n < \omega \;\&\; k \leq h(n)\}$ which collectively cover ω. Since h is assumed to be basic there is no problem building such a partition, for example one could use singletons. Label $J_n = \bigcup_{k \leq h(n)} J_{n,k}$. Then for each basic $f \in \omega^\omega$ let $f' : \omega \to \omega^{<\omega}$ be the function defined by $f'(n) = f \upharpoonright J_n$. More generally let $\mathcal{J} = \{f : \omega \to \omega^{<\omega} \mid \mathrm{dom}(f(n)) = J_n\}$. Notice that the basic elements of $\mathcal{J}$ are exactly $\{f' \mid f \in \omega^\omega \;\&\; f \sqsubseteq 0\}$ since from any f' we can build f and vice versa (by the the fact that the J_n's are basic). But now since the f''s are basic and each one codes a real one can by applying 2 plus some simple coding to find an h-slalom, $\sigma : \omega \to (\omega^{<\omega})^{<\omega}$ such that for every $n \in \omega$ $|\sigma(n)| \leq h(n)$ and $\sigma(n)$ is a set of finite partial functions from J_n to ω and for every basic $f' \in \mathcal{J}$ there are infinitely many $n \in \omega$ such that $f'(n) \in \sigma(n)$.

Let me denote $\sigma(n) = \{w^n_1, ..., w^n_{h(n)}\}$. Now set $g_n = \bigcup_{k \leq h(n)} w^n_k \upharpoonright J_{n,k}$ and let $g = \bigcup_{n<\omega} g_n$. Notice that this gives an element of ω^ω since the $J_{n,k}$'s were disjoint and collectively covered ω. I claim that g is as needed. Clearly g is reducible to the $J_{n,k}$'s, which are basic, and the w^n_k's, which are reducible to σ so g is reducible to σ. It remains to see that it is an infinitely-often-equal real. To see this, let $f \in \omega^\omega$ be basic and fix some n such that $f'(n) \in \phi(n)$ (recall that there are infinitely many such n). Notice that since $f'(n) \in \phi(n)$ there must be some $k \leq h(n)$ such that $f \upharpoonright J_m = w^n_k$. Now let $x_n \in J_{n,k}$ (recall that this set is assumed to be non-empty). We have that $f(x_n) = w^n_k(x_n) = g(x_n)$. But there are infinitely many such n and hence infinitely many such x_n so this completes the proof. □

A similar proof produces the last inclusion, $\mathcal{B}(\neq^*) \subseteq \mathcal{D}(\in^*)$. In words this inclusion states that any real which can build a real which is eventually different from all basic reals can build a real which is not eventually captured by any given slalom. I will prove the following more general lemma, whose statement and proof is inspired by [1], Theorem 2.2. Given an h-slalom σ and a function f let me say that f is *eventually never captured by* σ if there is some k such that for all $l > k$ $f(l) \notin \sigma(l)$.

Lemma 2.10. *For any real f, the following are equivalent.*

1. *The real f is eventually different from all basic reals.*
2. *The real f is such that for all basic reals h and all basic h-slaloms σ for all but finitely many $n \in \omega$ $f(n) \notin \sigma(n)$.*

Therefore $x \in \mathcal{B}(\neq^*)$ if and only if x builds a real which is eventually never captured by any basic h-slalom for any basic h.

Let me note before I prove Lemma 2.10 that it proves the inclusion $\mathcal{B}(\neq^*) \subseteq \mathcal{D}(\in^*)$ and hence Theorem 2.8. To see why, suppose that $x \in \mathcal{B}(\neq^*)$ and, without loss of generality suppose that x itself is a real which is eventually different from all basic reals. Then by the lemma x is eventually never captured by any basic slalom so, in particular for infinitely many n $x(n) \notin \sigma(n)$ for all basic σ, which means $x \in \mathcal{D}(\in^*)$.

Proof of Lemma 2.10. Fix some $f \in \omega^\omega$. The backward direction of this lemma is easy: if f is eventually never captured by any basic h-slalom for any basic h then in particular it is eventually never captured by the slalom sending $n \mapsto \{g(n)\}$ for each basic g and hence it is eventually different from each basic g.

For the forward direction, assume f is eventually different from all basic functions. Fix a basic h and, like in the proof of Lemma 2.9, in a basic fashion partition ω into finite, disjoint, non-empty sets $\{J_{n,k} \mid k \leq h(n)\}$. Let $J_n = \bigcup_{k \leq h(n)} J_{n,k}$. Let $f' : \omega \to \omega^{<\omega}$ be the function defined by $f'(n) = f \restriction J_n$. Then if σ is any basic h-slalom, let σ' be such that on input n gives $h(n)$ many finite partial fuctions $w_1^n, ..., w_{h(n)}^n$ with domain J_n where for all $k \leq h(n)$ and $l \in J_n$ $w_k^n(l)$ is the k^{th} greatest number in the set $\sigma(l)$ if such exists and 0 (say) otherwise. Suppose now towards a contradiction that there is a basic h-slalom σ such that $f(n) \in \sigma(n)$ for infinitely many n. For each n let $\sigma'(n) = \{w_1^n, ..., w_{h(n)}^n\}$. Then define $g_n = \bigcup_{k \leq h(n)} w_k^n \restriction J_{n,k}$ and let $g = \bigcup_{n<\omega} g_n$. Clearly g can built using σ, the function h and the $J_{n,k}$'s each of which is basic so g is basic. Thus there is a k such that for all $n > k$ we have that $f(n) \neq g(n)$. But, since there are infinitely many n such that $f(n) \in \sigma(n)$, there are infinitely many $n > k$ such that $f(n) \in \sigma(n)$ and therefore it follows that similarly we must have that there are infinitely many $n > k$ such that $f'(n)$ agrees with some w_j^n on some element of their shared domain for some $j \leq h(n)$. But this means $f(k) = g(k)$ for some $k \in J_{n,j}$ for infinitely many n's and j's which is a contradiction. □

Since this was the final inclusion to prove, Theorem 2.8 is now proved as well. □

Thus, even in this broad context one can construct diagrams for a wide variety of reduction concepts and a correspondence starts to form with the Cichoń diagram. This extends the proof given in the case of Turing degrees in [4] and gives a good framework for investigations into various computability reduction concepts. What it does not show, however, is that any of these nodes are non-empty or that the inclusions are strict. Indeed this is not necessarily the case. For instance $\mathcal{B}_{\leq_T}(\in^*) = \mathcal{B}_{\leq_T}(\neq^*)$ (see [4]). This is because, by a theorem of Rupprecht, the set $\mathcal{B}_{\leq_T}(\in^*)$ is simply the high reals, which as I mentioned above is also $\mathcal{B}(\leq^*)$. The analogue of this fact in the case of the classical Cichoń diagram is false since $\text{add}(\mathcal{N})$, the analogue of $\mathcal{B}_{\leq_T}(\in^*)$, can consistently be less than $\mathfrak{b}$, the analogue of $\mathcal{B}(\leq^*)$. The authors of [4] take this as evidence that the $\leq_T$-Cichoń diagram provides "only an analogy, not a full duality" [4, p. 3] with the classical Cichoń diagram. Theorem 2.8 proves the existence of a wide variety of such diagrams, therefore raising the question in each case of how strong the analogy between the reduction diagram and the classical diagram is, and whether we ever get a full duality. This depends on the strength of the reduction since, while the $\leq_T$ diagram gives only an analogy, I show in the next section that in the $\leq_W$ diagram the inclusions proved in Theorem 2.8 constitute the only ones true in every model of ZFC, thereby suggesting something closer to a true duality.

3 Separations in the $\leq_W$-Cichoń Diagram

From now on fix an inner model W. I work in the language of set theory with an extra predicate for W and the theory ZFC(W), that is ZFC with replacement and comprehension holding for formulas containing W. I view $W = L$ as a central case but it turns out that the analysis works out the same for arbitrary W.

Before presenting the full $\leq_W$-Cichoń diagram, let me state clearly what the unbounding and dominating sets are for the combinatorial relations defined in the last section for $\leq_W$.

1. $\mathcal{B}(\in^*)$ is the set of reals x such that there is a slalom $\sigma \in W[x]$ that eventually captures all reals in W.
2. $\mathcal{B}(\leq^*)$ is the set of reals x such that there is a real $y \in W[x]$ that eventually dominates all reals in W. These are sometimes called *dominating reals* (for W).
3. $\mathcal{B}(\neq^*)$ is the set of reals x such that there is a real $y \in W[x]$ that is eventually different from all reals in W. These are sometimes called *eventually different reals* (for W).
4. $\mathcal{D}(\in^*)$ is the set of reals x such that there is a real $y \in W[x]$ that is not eventually captured by any slalom in W.
5. $\mathcal{D}(\leq^*)$ is the set of reals x such that there is a real $y \in W[x]$ that is not eventually dominated by any real in W. These are sometimes called *unbounded reals* (for W).

6. $\mathcal{D}(\neq^*)$ is the set of reals x such that there is a real $y \in W[x]$ that is equal infinitely often to every real in W. These are sometimes called *infinitely-often-equal reals* (for W).

In this section I will study how a variety of known forcing notions over W can create separations in the $\leq_W$-Cichoń diagram as described in the previous section. Of course ZFC(W) cannot prove any separations since if $V = W$ or, more generally V and W have the same reals, every node in the $\leq_W$-diagram will be empty. However, using simple forcings I will show that one can produce a wide variety of possible constellations for the $\leq_W$-diagram. The main theorem of this section is the following.

Theorem 3.1. *The Cichoń diagram for $\leq_W$ as described in the previous section is complete for ZFC(W)-provable implications. In other words if A and B are two nodes in the diagram and there is not an arrow from A to B in the $\leq_W$-diagram then there is a forcing extension of W where $A \not\subseteq B$.*

That these implications all hold follows from Theorem 2.8 since $\leq_W$ extends $\leq_T$.

$$\begin{array}{ccccc}
 & & \mathcal{B}_{\leq_W}(\neq^*) & \longrightarrow & \mathcal{D}_{\leq_W}(\in^*) & \longrightarrow & \omega^\omega \setminus (\omega^\omega)^W \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{B}_{\leq_W}(\leq^*) & \longrightarrow & \mathcal{D}_{\leq_W}(\leq^*) & & \\
 & & \uparrow & & \uparrow & & \\
\emptyset & \longrightarrow & \mathcal{B}_{\leq_W}(\in^*) & \longrightarrow & \mathcal{D}_{\leq_W}(\neq^*) & &
\end{array}$$

Let me note one word on the relation between my diagram and the standard Cichoń diagram as commonly studied, for example in [2]. Here I have focused on the so-called combinatorial nodes as discussed by [4]. As noted in the introduction I view this diagram in correspondence with the classical one via the mapping sending unbounded or dominating families with respect to a certain relation to the sets of reals x such that in $W[x]$ the reals of W are not unbounded or dominating. I have included this fragment of the Cichoń diagram to make this analogy clear visually.

$$\begin{array}{ccccccc}
 & & \mathrm{non}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{N}) & \longrightarrow & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & \\
 & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & \\
 & & \uparrow & & \uparrow & & \\
\aleph_1 & \longrightarrow & \mathrm{add}(\mathcal{N}) & \longrightarrow & \mathrm{cover}(\mathcal{M}) & &
\end{array}$$

Figure 4: The Combinatorial Nodes of the Standard Cichoń Diagram

The details of these correspondences for $\leq_T$ can be found in [4] and similar ideas hold in the present case. In a planned sequel [11] I will treat the mising nodes, namely those corresponding to invariants of measure and category.

3.1 Sacks Forcing

The first forcing I will look at is Sacks forcing, $\mathbb{S}$. Recall that conditions in $\mathbb{S}$ are perfect trees $T \subseteq 2^{<\omega}$ ordered by inclusion. If G is $\mathbb{S}$-generic then the unique branch in the intersection of all members of G is called a *Sacks real*. I denote such a real s.

Theorem 3.2. *In the Sacks extension all nodes of $\leq_W$-Cichoń diagram other than $\omega^\omega \setminus (\omega^\omega)^W$ are empty. Moreover there are further forcings $\mathbb{P}_1$ and $\mathbb{P}_2$ over V such that this cut and only this cut is realized and $V^{\mathbb{P}_1} \models CH$ and $V^{\mathbb{P}_2} \models \neg CH$.*

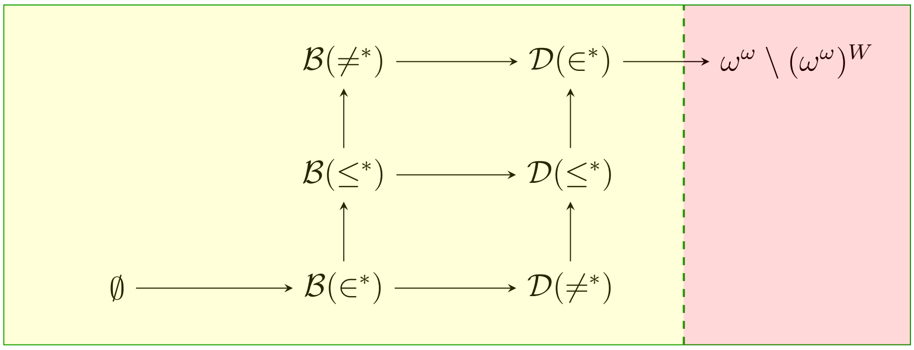


Figure 5: After Sacks forcing

Proof. Recall that Sacks reals have the following useful and well known property called the *Sacks property*[2] [cf. [3]]: if $f : \omega \to V$ is a function in $V[s]$ then there is a function $g : \omega \to V$ in V such that for all n, $f(n) \in g(n)$ and $|g(n)| \leq n$. As a result, all reals added by $\mathbb{S}$ and hence all reals in V that are not in W can be captured by a slalom from the ground model i.e. W. Thus, $W[s]$ thinks that $\mathcal{D}(\in^*)$ is empty but $s \in \omega^\omega \setminus (\omega^\omega)^W$ and hence the only non-empty set in the $\leq_W$-Cichoń diagram is the latter.

For the second part of the theorem, let $\mathbb{Q}$ be the countably closed forcing to collapse the continuum to $\aleph_1$. Since this forcing is countably closed it adds no new reals so in $V^{\mathbb{Q}}$ the $\leq_W$-diagram remains unchanged and CH is true. Thus $\mathbb{Q}$ is the $\mathbb{P}_1$ required. For $\mathbb{P}_2$, recall that $\mathbb{S}$ is proper and assuming CH in the ground model, forming an $\aleph_2$ length iteration of $\mathbb{S}$ with countable support will produce a model of $2^{\aleph_0} = \aleph_2$ in which the Sacks property still holds. Thus if $\mathbb{P}$ is such an iteration, then forcing with $\mathbb{P}_2 = \mathbb{Q} * \dot{\mathbb{P}}$ will give the required model. $\square$

[2] In the proof of this fact, one usually obtains that $|g(n)| \leq 2^n$ but as is noted in [3, pg. 86] any function tending towards infinity works. Since $n \mapsto n$ is relevant here, I have reformulated the Sacks property with this bound.

Notice that this proof actually gives a more general result: instead of forcing over W one can force over V. In this case the same proof shows that if $W \subseteq V$ and s is V-generic for $\mathbb{S}^W$, then all the nodes in the $\leq_W$-diagram in $V[s]$ agree with V except the top right. Similar phenomena will hold in the proceeding cases.

3.2 Cohen Forcing

Let $\mathbb{C} = Add(\omega, 1)$ be the forcing to add one Cohen real. The main theorem of this section is:

Theorem 3.3. *Let c be a Cohen real generic over W. Then in $W[c]$ the following hold:*

1. $\emptyset = \mathcal{B}(\in^*) = \mathcal{B}(\leq^*) = \mathcal{B}(\neq^*)$
2. $\mathcal{D}(\neq^*) = \mathcal{D}(\leq^*) = \mathcal{D}(\in^*) = \{x \mid \exists c \in W[x] \text{ Cohen over } W\} = \omega^\omega \setminus (\omega^\omega)^W$

Thus, the full diagram for Cohen Forcing is:

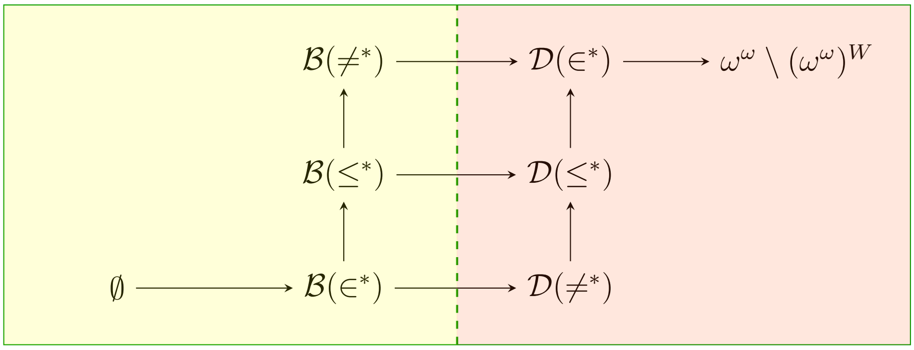


Figure 6: After Cohen forcing

Proof. There are two parts to this proof. First I need to show that all of the elements on the left are empty. Since $\mathcal{B}(\in^*) \subseteq \mathcal{B}(\leq^*) \subseteq \mathcal{B}(\neq^*)$ it suffices to show that Cohen forcing adds no reals which are eventually different from all ground model reals. This is a standard argument but I repeat it here for completeness. Let $\{p_i\}_{i\in\omega}$ enumerate the conditions of $\mathbb{C}$ and suppose that $\Vdash_{\mathbb{C}} \dot{f} : \omega \to \omega$. Then, for each i, pick a $q_i \leq p_i$ which decides the value of $\dot{f}(i)$ in other words let $q_i \Vdash \dot{f}(\check{i}) = \check{j}_i$ for some j_i. Now, in the ground model, set $g(i) = j_i$. Finally, suppose for contradiction that there was a $k \in \omega$ and a $p \in \mathbb{C}$ such that $p \Vdash \forall l > \check{k}\ \check{g}(l) \neq \dot{f}(l)$. But then one can find an $i > k$ and a q_i such that $q_i \Vdash \check{g(i)} = \dot{f}(i)$, which is a contradiction.

So Cohen forcing leaves the left side of the diagram trivialized. The right side however changes since it's dense for c to equal every real in W infinitely often so $c \in \mathcal{D}(\neq^*)$. The second part of the proof is to show that every real added by Cohen forcing adds an element to $\mathcal{D}(\neq^*)$. Since $\mathcal{D}(\neq^*) \subseteq \mathcal{D}(\leq^*) \subseteq \mathcal{D}(\in^*) \subseteq \omega^\omega \setminus (\omega^\omega)^W$ it suffices to show that, $\omega^\omega \setminus (\omega^\omega)^W \subseteq \mathcal{D}(\neq^*)$. Let $x \in W[c] \setminus W$ be a new real and consider now the model $W[x]$.

By the intermediate model theorem it must be the case that $W[x]$ is a generic extension of W and that $W[c]$ is a generic extension of $W[x]$ so the forcing to add x is a non trivial factor of Cohen forcing so it must in fact be isomorphic to it. Thus in $W[x]$ there is a real d which is Cohen generic over W, and d is infinitely often equal to every real in W so $x \in \mathcal{D}(\neq^*)$. $\square$

3.3 Hechler Forcing

Let $\mathbb{D}$ be Hechler forcing and let d be the associated dominating real. Recall that conditions of $\mathbb{D}$ are pairs $(p, \mathcal{F})$ where p is a finite partial function from ω to ω and $\mathcal{F}$ is a finite family of elements of ω^ω. The order is given by $(q, \mathcal{G}) \leq_{\mathbb{D}} (p, \mathcal{F})$ if and only if $q \supseteq p$, $\mathcal{G} \supseteq \mathcal{F}$ and for all $n \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$ and all $f \in \mathcal{F}$, $q(n) > f(n)$. Note that since d is dominating, $d \in \mathcal{B}(\leq^*)$.

Theorem 3.4. *After Hechler forcing over W the $\leq_W$-diagram has*

1. $\emptyset = \mathcal{B}(\in^*)$,
2. $\mathcal{B}(\leq^*) = \mathcal{B}(\neq^*)$ *and*
3. $\mathcal{D}(\neq^*) = \mathcal{D}(\leq^*) = \mathcal{D}(\in^*) = \omega^\omega \setminus (\omega^\omega)^W$.

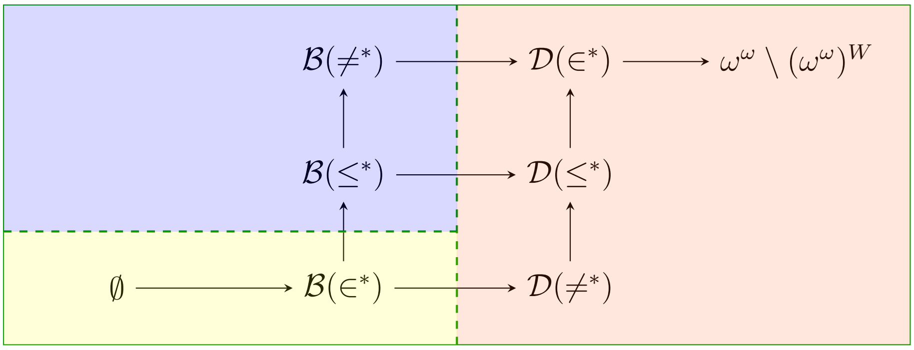


Figure 7: After Hechler forcing

The proof of this theorem is broken up into several lemmas. First I show that $\mathbb{D}$ adds no slaloms eventually capturing all the ground model reals.

Lemma 3.5. *After Hechler forcing over W the set $\mathcal{B}(\in^*)$ is empty.*

Proof. Let me begin with a simple observation about Hechler forcing: if σ is a sentence in the forcing language and p is the stem of a condition (the first coordinate) then it cannot be that there are there are finite families of functions $\mathcal{F}$ and $\mathcal{G}$ such that $(p, \mathcal{F}) \Vdash \sigma$ and $(p, \mathcal{G}) \Vdash \neg\sigma$. To see why, simply notice that $(p, \mathcal{F} \cup \mathcal{G})$ is a condition extending them both. Now, using the weak homogeneity of Hechler forcing, suppose that

$\Vdash_{\mathbb{D}}$"$\dot{s}$ is a slalom eventually capturing all elements of $(\omega^\omega)^W$". Now fix an enumeration of $\omega^{<\omega} = \{p_0, p_1, p_2, ...\}$ and consider the following function $f : \omega \to \omega$ such that $f(n) = \sup \{k \mid \exists i < n \, \exists \mathcal{F} \, (p_i, \mathcal{F}) \Vdash \check{k} \in \dot{s}(n)\} + 1$. Note that f is definable in W.

Claim 3.6. *The function f is total and well defined.*

Proof. To see this, notice that since the maximal condition forces that $\dot{s}$ names a slalom, all conditions force that for all n, $\dot{s}(n)$ has size at most n. In particular, no condition can force more than n check names to be in $s(n)$. Moreover, by the simple observation I began with, there cannot be more than n check names forced to be in $\dot{s}$ by any set of conditions sharing the same stem. Thus, since there are only finitely many stems being considered, each of which can only be paired to force at most n check names, there are at most n^2 numbers in the set $\{k \mid \exists i < n \, \exists \mathcal{F} \, (p_i, \mathcal{F}) \Vdash \check{k} \in \dot{s}(n)\}$ so f is well defined and always finite. □

Now work in $W[d]$. It remains to show that f is not eventually captured by the slalom $s = \dot{s}_d$. Suppose not and let $k, j \in \omega$ such that $(p_j, \mathcal{F}) \Vdash \forall l > \check{k} \; \check{f(k)} \in \dot{s}(\check{k})$. Let now let $l > k, k$ be such that $(p_l, \mathcal{G}) \leq (p_j, \mathcal{F})$. Then, $(q, \mathcal{G}) \Vdash \check{f(l)} \in \dot{s}(l)$ but this implies $f(l) \geq f(l) + 1$, which is a contradiction. □

Continuing, recall the following theorem of Brendle and Löwe. I have adapted it to our specific situation and terminology:

Theorem 3.7. *([5, Corollary 13]) In $W[d]$, if $f \in \omega^\omega$ is eventually different from all of W's reals, then it eventually dominates them.*

Therefore all the reals in $W[d]$ in $\mathcal{B}(\neq^*)$ are automatically in $\mathcal{B}(\leq^*)$. As an immediate corollary the following is true.

Corollary 3.8. *In the extension of W by a Hechler real, $\mathcal{B}(\neq^*) = \mathcal{B}(\leq^*)$.*

Thus, we know what happens on the left side of the digram. For the right side of the diagram, the following fact is well known and easily verified:

Fact 3.9. *Let d be $\mathbb{D}$-generic over W. Then d mod 2 i.e. the parity of d is a Cohen generic over W.*

Therefore Hechler forcing adds Cohen reals. Indeed, since $\mathbb{C} * \dot{\mathbb{C}}$ is forcing equivalent to $\mathbb{C}$, by the intermediate model theorem $\mathbb{D}$ can be decomposed in to $\mathbb{C} * \mathbb{Q}$ where $\mathbb{Q}$ is some quotient forcing. But then $\mathbb{D} \cong \mathbb{C} * \dot{\mathbb{Q}} \cong \mathbb{C} * \dot{\mathbb{C}} * \dot{\mathbb{Q}} \cong \mathbb{C} * \dot{\mathbb{D}}$. So Hechler forcing is the same as Cohen forcing, followed by Hechler forcing. The classification of subforcings of Hechler forcing is a very interesting, but somewhat delicate topic due in part to subtle differences in a variety of different "Hechler Forcings". Palumbo [10] has solved this problem completely assuming there is a proper class of Woodin cardinals. I do not know of a full solution in the general case.

Summarizing, we have seen so far that there are at least three different sets present in the diagram for a Hechler real: $\emptyset$ for $\mathcal{B}(\in^*)$, the dominating reals, for $\mathcal{B}(\leq^*) = \mathcal{B}(\neq^*)$

and the reals that add Cohen real, which by the previous section, are all included in $\mathcal{D}(\neq^*)$ and thus the entire right column. To finish the analysis, I use the following fact, due to Palumbo.

Fact 3.10. *([10, Theorem 8.1]) Let d be $\mathbb{D}$-generic over W and let M be an intermediate model i.e. $W \subseteq M \subseteq W[d]$. Then if $M \neq W$, there is a real $x \in M$ which is Cohen-generic over W.*

Using Fact 3.10 I can now show the following:

Corollary 3.11. *In the extension of W by a Hechler real, all the new reals construct a real which is equal to the reals in W infinitely often i.e. $\omega^\omega \setminus (\omega^\omega)^W = \mathcal{D}(\neq^*)$.*

Proof. Let $x \in W[d] \setminus W$ be a real. Then by Fact 3.10 there is $\mathbb{C}$-generic real over W in $W[x]$ so $x \in \mathcal{D}(\neq^*)$. $\square$

Notice that this completely determines the diagram for a Hechler real. Since $\mathcal{D}(\neq^*)$ is all new reals, every node in the diagram is a subset of it. Thus all nodes on the right side are equal, $\mathcal{B}(\in^*)$ is empty and $\mathcal{B}(\leq^*) = \mathcal{B}(\neq^*)$ form a proper subset of the $\mathcal{D}$'s. This finishes the proof of Theorem 3.4

3.4 Eventually Different Forcing

Let $\mathbb{E}$ be *eventually different forcing*, which is defined like $\mathbb{D}$ except that stems of extensions need simply be eventually different from the reals in the second component, not dominating. I will show that:

Theorem 3.12. *Assume that every set of reals in $L(\mathbb{R})$ has the Baire property (this is implied by sufficiently large cardinals). Let e be an $\mathbb{E}$-generic real over W. Then in $W[e]$ the following hold:*

1. $\mathcal{B}(\in^*) = \mathcal{B}(\leq^*) = \emptyset$,
2. $\mathcal{B}(\neq^*) \subsetneq \mathcal{D}(\neq^*) = \mathcal{D}(\leq^*) = \mathcal{D}(\in^*) = \omega^\omega \setminus (\omega^\omega)^W$.

Thus in particular the full diagram for eventually different forcing is as shown in Figure 7.

To prove this I will use a series of lemmas similar to those used in the case of Hechler forcing. First, a straightforward modification of Lemma 3.5 shows that there are no dominating reals in $W[e]$:

Lemma 3.13. $W[e] \models \mathcal{B}(\in^*) = \mathcal{B}(\leq^*) = \emptyset$

To complete the analysis of the $\leq_W$-diagram after forcing with $\mathbb{E}$, I need the analogy of Palumbo's Fact 3.10 for $\mathbb{E}$. Unfortunately, his argument uses a tree version of $\mathbb{D}$ that, as far as I can tell, is not available for $\mathbb{E}$. As such, I only know how to prove Palumbo's result for $\mathbb{E}$ assuming sufficient large cardinals. I conjecture that it should hold in ZFC.

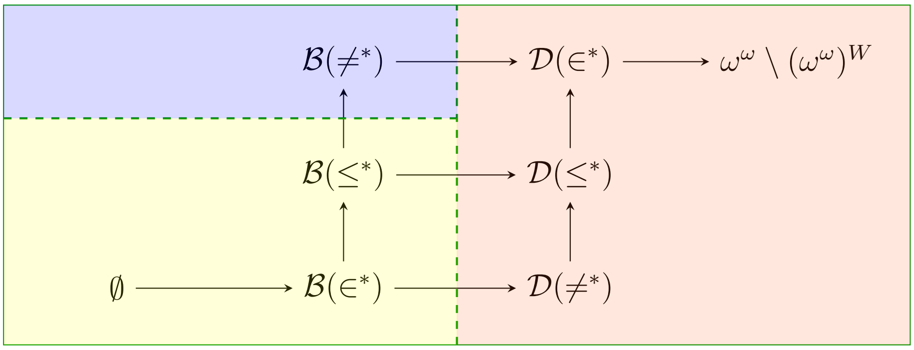


Figure 8: After Eventually Different forcing

Lemma 3.14. *Assume that every set of reals in $L(\mathbb{R})$ has the property of Baire. Then in every nontrivial intermediate model between W and $W[e]$ there is a real c which is $\mathbb{C}$-generic over W.*

A proof of this is sketched in [10, pg 38] for $\mathbb{D}$ but the reader will notice that it goes through equally well for $\mathbb{E}$. Indeed the centerpiece of the argument involves a fact, due to Shelah and Gitik [6, Proposition 4.3] that given any sufficiently well-defined σ-centered forcing $\mathbb{P}$, if certain filters of $\mathbb{P}$ in $L(\mathbb{R})$ have the property of Baire, then $\mathbb{P}$ will add a Cohen real. It is not hard to see from the combination of the Gitik-Shelah and the Palumbo arguments that "sufficiently well defined" includes all subforcings of $\mathbb{E}$. Thus, assuming all sets of reals have the property of Baire the result goes through.

Using this lemma, by the same argument given for $\mathbb{D}$, we have the proof of Theorem 3.12.

The use of large cardinals here is unfortunate and I hope it can be improved on. Let me note however that even without large cardinals I have shown that there is a model realizing the cut determined by $\mathcal{B}(\in^*) = \mathcal{B}(\neq^*) = \emptyset$.

3.5 Localization Forcing

In this section I study *Localization forcing*, the forcing to add a generic slalom capturing all ground model reals.

Definition 3.15 (Localization Forcing (cf [5]))**.** The localization forcing $\mathbb{LOC}$ is defined as the set of pairs $(s, \mathcal{F})$ such that $s \in ([\omega]^{<\omega})^{<\omega}$ is a finite sequence with $|s(n)| \leq n$ for all $n < |s|$ and $\mathcal{F}$ is a a finite family of functions in Baire space with $|\mathcal{F}| \leq |s|$. The order is $(t, \mathcal{G}) \leq_{\mathbb{LOC}} (s, \mathcal{F})$ if and only if $t \supseteq s$, $\mathcal{G} \supseteq \mathcal{F}$ and $f(n) \in t(n)$ for all $f \in \mathcal{F}$ and all $n \in |t| \setminus |s|$. We think of the first component as a finite approximation to a slalom we are trying to build and as such I will often refer to the length of the sequence as its "domain" and write $\mathrm{dom}(s)$.

Unfortunately I do not have a full characterization of the diagram in the case of $\mathbb{LOC}$. The following theorem summarizes the state of knowledge.

Theorem 3.16. *Let σ be a slalom which is $\mathbb{LOC}$-generic over W. Then in $W[\sigma]$ all the nodes in the diagram are non-empty and we have that $\mathcal{B}(\in^*)$ is a proper subset of $\mathcal{B}(\leq^*)$ and $\mathcal{D}(\neq^*)$. Also $\mathcal{B}(\leq^*) \subsetneq \mathcal{B}(\neq^*)$. In particular, Figure 9 is a partial diagram for $\mathbb{LOC}$.*

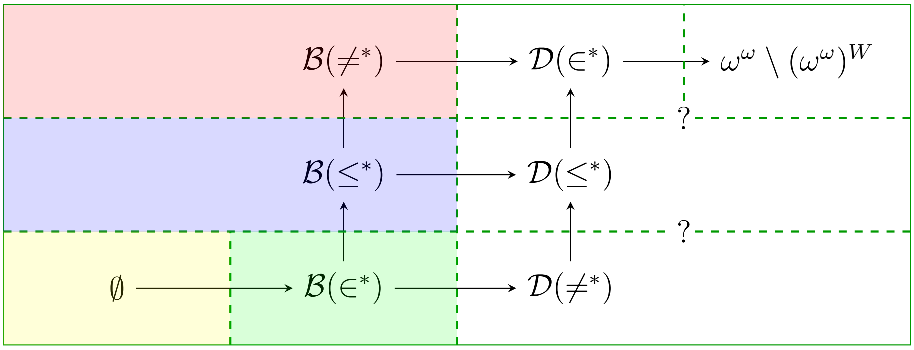


Figure 9: Partial diagram after Localization forcing

Proving this theorem amounts to showing that $\mathbb{LOC}$ adds both $\mathbb{D}$ and $\mathbb{E}$ generics. I start with $\mathbb{D}$. Notice first that $\mathbb{LOC}$ adds a dominating real. Indeed if σ is a generic slalom in $W^{\mathbb{LOC}}$ then $d(n) := \max \sigma(n)$ has this property. This is actually a Hechler real:

Lemma 3.17. *Let $\sigma \in W^{\mathbb{LOC}}$ be a generic slalom eventually capturing all ground model reals. Then, $d(n) := \max \sigma(n)$ is $\mathbb{D}$-generic over W.*

To prove this I will need a simplified version of $\mathbb{D}$: in the first component of a condition I will assume that the domain is a finite initial segment of ω and instead of having the second component of a condition of $\mathbb{D}$ be a finite family of functions, it will be a single function. Then $(q, g) \leq_{\mathbb{D}} (p, f)$ if and only if q extends p, for all $n \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$, $q(n) \geq f(n)$ and for all $n \in \omega$, and $g(n) \geq f(n)$. It's not hard to see that this version of $\mathbb{D}$ is forcing equivalent to the original one I defined.

Proof. Recall that a *projection* $\pi : \mathbb{P} \to \mathbb{Q}$ between two posets is an order preserving map which sends the maximal element of $\mathbb{P}$ to the maximal element of $\mathbb{Q}$ and for all $p \in \mathbb{P}$ and all $q \leq \pi(p)$ there is some $\bar{p} \leq p$ such that $\pi(\bar{p}) \leq q$. If a projection exists between $\mathbb{P}$ and $\mathbb{Q}$ then the image $\pi''G$ of a $\mathbb{P}$-generic filter generates a $\mathbb{Q}$-generic filter. Therefore to prove the lemma it suffices to show that the map $\pi : \mathbb{LOC} \to \mathbb{D}$ such that $\pi(s, \mathcal{F}) = (n \mapsto \max s(n), \Sigma\mathcal{F})$ where $\Sigma\mathcal{F}$ is the pointwise sum, is a projection. To see why, note that if $(s, \mathcal{F}) \in \mathbb{LOC}$ and let, for all $n \in \mathrm{dom}(s)$, $p(n) = \max s(n)$ and let $f = \Sigma\mathcal{F}$. Since $\mathcal{F}$ is finite this is well defined. Then the pair (p, f) is a $\mathbb{D}$ condition and the union of all conditions such defined from elements of the $\mathbb{LOC}$ generic defining σ is the d from the statement of the lemma.

It is routine to check that $\pi(1_{\mathbb{LOC}}) = 1_{\mathbb{D}}$ and that the map π is order preserving. The difficulty is in verifying the third condition of projections. To this end, let $(s, \mathcal{F}) \in \mathbb{LOC}$

and let $(p, f) = \pi(s, \mathcal{F})$. Let $(p', f') \leq (p, f)$ and let $D \subseteq \mathbb{D}$ be a set of conditions which is dense below (p', f'). It suffices to find a strengthening $(t, \mathcal{G})$ of $(s, \mathcal{F})$, such that $(n \mapsto \max t(n), \Sigma G) \in D$. To do this, first, find a function $g : \omega \to \omega$ such that for all $n \notin \mathrm{dom}(p)$, $g(n) > n + f$ and otherwise is at least as big as f. Then, (p, g) strengthens (p, f) and is compatible with (p', f'). Let $(q, h) \in D$ strengthen (p, g).

Now, we can build our new $\mathbb{LOC}$ condition. Define $H : \omega \to \omega$ by $H(n) = h(n) - f(n)$. Notice that since $g(n)$ was assumed to be bigger than $f(n)$ for all n and $h(n) \geq g(n)$ since it is a strengthening it follows that H is in fact always positive. Moreover, $f + H = \Sigma\mathcal{F} + H = h$. It remains to show that there is a $t \supseteq s$ such that $\mathrm{dom}(t) = \mathrm{dom}(q)$, for all $n \in \mathrm{dom}(t)$, $\max t(n) = q(n)$ and for all $n \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ and all $f \in \mathcal{F}$, $f(n) \in t(n)$. Once this has been done $(t, \mathcal{F} \cup \{H\})$ will be the desired condition. I claim that this is all possible. I will describe a t extending s be defined on the domain of q (by construction, the domain of q contains that of s). Without loss of generality $|\mathrm{dom}(q)| > |\mathrm{dom}(s)| + 2$. Thus, the domain of t will be large enough to accomodate the side condition $\mathcal{F} \cup \{H\}$. Let $|\mathcal{F}| = k$ and enumerate $\mathcal{F} = \{f_0, ..., f_{k-1}\}$. Note that $k < n$ for all $n \in \mathrm{dom}(q) \setminus \mathrm{dom}(s)$. Now, for each $n \in \mathrm{dom}(q) \setminus \mathrm{dom}(s)$, let me define $t(n)$. Notice first that one must put in all k numbers $\{f_0(n), ..., f_{k-1}(n)\}$ and we also want $\max t(n) = q(n)$ so add this in too. Since $n > k$, one needs to simply add $n - k - 1$ additional numbers $\{j_0, ..., j_{n-k-2}\}$ such that each one is less than $q(n)$ and different from all numbers in the set $\{f_0(n), ..., f_{k-1}(n), q(n)\}$. This is possible however, since by construction $q(n) \geq g(n)$ for all $n \notin \mathrm{dom}(p)$ and $g(n) > n + \Sigma_{i<k} f_i(n)$ on this domain. Thus, there must be at least n between the maximum of the $f_i(n)$'s and $q(n)$, which is more than we needed. □

Now, I show that $\mathbb{LOC}$ adds an $\mathbb{E}$-generic real. This fact was first told to me (without proof) in private communication with J. Brendle. I thank him for pointing it out to me.

Lemma 3.18. *The forcing* $\mathbb{LOC}$ *adds an* $\mathbb{E}$*-generic real.*

Proof. Given a condition $(s, \mathcal{F}) \in \mathbb{LOC}$ define a stem for an $\mathbb{E}$-condition as $p_s : \mathrm{dom}(s) \to \omega$ by letting for all $n \in \mathrm{dom}(s)$ $p_s(n)$ be equal to the k^{th} natural number m not in the set $s(n)$ where the pointwise sum $\Sigma s(n) \equiv k \bmod n$. We claim that the map $\pi : \mathbb{LOC} \to \mathbb{E}$ defined by $\pi(s, \mathcal{F}) = (p_s, \mathcal{F})$ is a projection. Clearly the maximal condition is sent to the maximal condition and this map is order preserving. Let $(s, \mathcal{F}) \in \mathbb{LOC}$, and let $(q, \mathcal{G}) \leq_{\mathbb{E}} (p_s, \mathcal{F})$. We need to show that there is a strengthening of $(q, \mathcal{G})$ in the image of π. To this end, note that we can assume with out loss that $|\mathcal{G}| < \mathrm{dom}(q)$ since otherwise we can strengthen to make this true. Now, define a partial slalom as follows: $s_q : \mathrm{dom}(q) \to [\omega]^{<\omega}$. For $n \in \mathrm{dom}(p)$ let $s_q(n) = s(n)$. For $n \notin \mathrm{dom}(p)$ let $q(n) = m$ and suppose that m is the k^{th} not in $\{f(n) \mid f \in \mathcal{F}\}$ and suppose that this set has size $l < n$ (the $<$ follows from the fact that $(p, \mathcal{F})$ is in the image of π). Then, pick $n - l$ numbers $m_l, m_{l+1}, ..., m_{n-1}$ all greater than every $f(n)$ for $f \in \mathcal{F}$ and not equal to m so that $\Sigma_{f \in \{} f(n) + \Sigma_{i=l}^{n-1} m_i \equiv k \bmod n$. This can be accomplished, for instance, as follows: if $\Sigma_{f \in \mathcal{F}} f(n) \equiv j \bmod n$ then let $m_l \equiv k - j \bmod n$ greater than all the $f(n)$'s and let all other m_i's be multiples of n. Finally let $s_q(n) = \{f(n) \mid f \in \mathcal{F}\} \cup \{m_l, ..., m_{n-1}\}$. Then $(s_q, \mathcal{G}) \leq (s, \mathcal{F})$ and $\pi(s_q, \mathcal{G}) = (q, \mathcal{G})$ as needed. □

Combining all of these results then proves Theorem 3.16 since both $\mathbb{D}$ and $\mathbb{E}$ add Cohen reals realizing the split down the middle in Figure 9.

As an aside notice that there seem to be other eventually different reals added by $\mathbb{LOC}$:

Observation 3.19. *Let $\sigma \in W^{\mathbb{LOC}}$ be a generic slalom eventually capturing all ground model reals. Let $a(n)$ be defined as the least $k \notin \sigma(n)$. Then a is a real which is eventually different from all ground model reals but is not an $\mathbb{E}$-generic real.*

Proof. First notice that the a described in the theorem is in fact eventually different from all ground model reals since every real eventually is captured by σ and after that point a is different from it. Moreover, notice that a is not only not dominating over the ground model reals but actually not even unbounded since, given any real $f \in W$ growing faster than the identity ($n \mapsto n + 2$ even), the least k not in $\sigma(n)$ must be less than $f(n)$ since $|\sigma(n)| = n$. From this it follows that a is not an $\mathbb{E}$-generic since it is not unbounded. □

This lemma is somewhat surprising and indeed I do not know exactly what the forcing adding the real a is or if it is a previously studied notion.

3.6 Random Real Forcing

I denote random real forcing by $\mathbb{B}$. The diagram for random real forcing is as described in the theorem below.

Theorem 3.20. *Let r be a random real over W. Then in $W[r]$ the $\leq_W$-Cichoń diagram is determined by the separations $\mathcal{B}(\in^*) = \mathcal{B}(\leq^*) = \mathcal{D}(\neq^*) = \mathcal{D}(\leq^*) = \emptyset$ and $\mathcal{B}(\neq^*) = \mathcal{D}(\in^*) = \omega^\omega \setminus (\omega^\omega)^W$.*

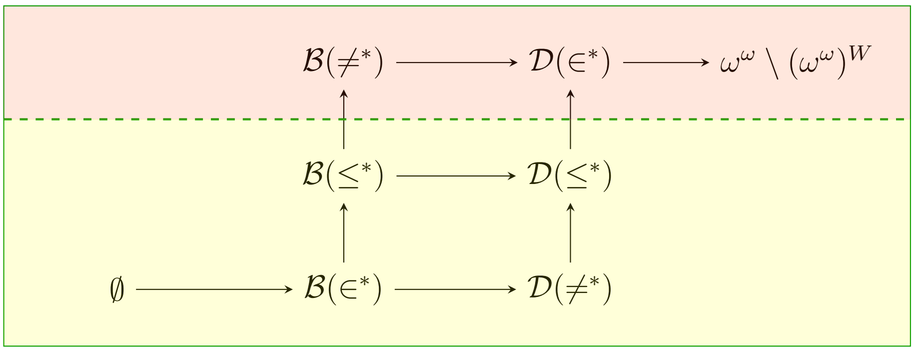


Figure 10: After Random Real forcing

The proof of this theorem follows from the following list of facts that are well known and can be found in [2], Chapter 3.

Fact 3.21. *The random real forcing* $\mathbb{B}$

1. *Adds no unbounded reals,*
2. *Adds an eventually different real and*
3. *If* $x \in W[r] \cap \omega^\omega \setminus W \cap \omega^\omega$ *then there is a real which is random over* W *in* $W[x]$.

Proof of Theorem 3.20. Since by 1 of Fact 3.21, $\mathbb{B}$ adds no unbounded reals $\mathcal{D}(\leq^*)$ is empty. Now, suppose $x \in W[r] \setminus W$, then there is a $y \leq_W x$ which is also random over W by 3 of Fact 3.21. Thus by 2 of Fact 3.21 we get that $x \in \mathcal{B}(\neq^*)$. Therefore $\omega^\omega \setminus (\omega^\omega)^W \subseteq \mathcal{B}(\neq^*)$ and the result follows. □

3.7 Laver Forcing

Let me now turn to Laver forcing, $\mathbb{L}$. Recall that conditions in Laver forcing are trees $T \subseteq \omega^{<\omega}$ with a distinguished *stem*, that is, a linearly ordered initial segment, after which there is infinite branching at each node. The order is inclusion. The union of the stems of the trees in a generic for $\mathbb{L}$ form a real, called a Laver real. Let l denote such a real over W. Recall that l is dominating. The main theorem of this section is

Theorem 3.22. *Let* l *be a Laver real over* W. *Then the* $\leq_W$ *diagram in* $W[l]$ *has* $\emptyset = \mathcal{B}(\in^*) = \mathcal{D}(\neq^*)$ *and all other nodes are equal to the set of all new reals.*

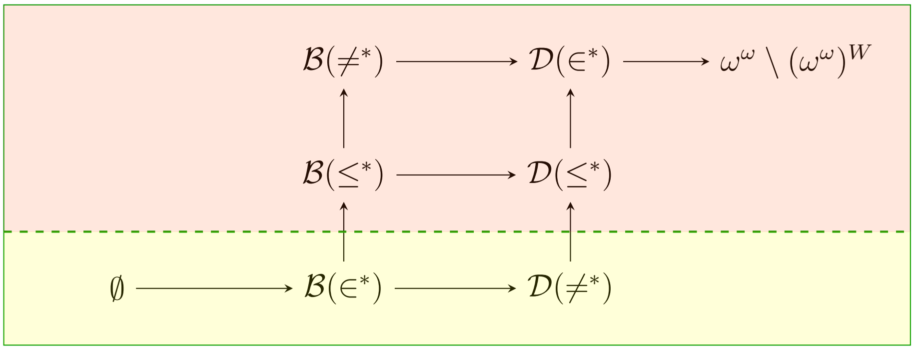


Figure 11: After Laver forcing

To prove this theorem I begin by noting that since l is dominating the following holds.

Fact 3.23. $W[l] \models \mathcal{B}(\leq^*) \neq \emptyset$

In fact, more is true, for unlike $\mathbb{D}$, $\mathbb{L}$ does not add Cohen reals. Indeed it does not add any reals in $\mathcal{D}(\neq^*)$.

Lemma 3.24. *If l is a Laver real over W then in $W[l]$ there are no infinitely-often-equal reals for W. In other words, $\mathcal{D}(\neq^*) = \emptyset$.*

Proof. [3] It is a standard fact about $\mathbb{L}$ that it satisfies what is known as *the Laver property* (see [2, Chapter 6]) which states that any real in the extension $W[l]$ which is bounded by a ground model real can eventually be captured in an h-slalom for any real $h \in W$ whose lim sup is infinity. Thus, if $f \in W[l]$ is an infinitely-often-equal real over W then for any $h \in (\omega^\omega)^W$ tending towards infinity the real $f' = \min\{f, h\}$ is bounded by h and infinitely-often-equal to every real which eventually grows slower than h. However, by the Laver property it follows that one can find a g-slalom $\sigma \in W$ for some $g \in W$ such that for all n $g(n) < h(n)$ and f' is eventually captured by σ so there cannot be infinitely many n such that $f'(n) = \max \sigma(n) + 1$ but this will grow slower than h, which is a contradiction. □

I will also need the following result due to Groszek [7]:

Fact 3.25. *(Groszek, essentially [7, Theorem 7]) Laver reals satisfy the following minimality property: if x is a real such that $x \in W[l] \setminus W$ then $l \in W[x]$.*

Now I can prove the separations in the Laver extension.

Proof of Theorem 3.22. By what I have shown, it follows that $\mathcal{B}(\in^*) = \mathcal{D}(\neq^*) = \emptyset$. Moreover, since l is dominating and minimal, it follows that $\mathcal{B}(\leq^*)$ must be all the new reals added by the forcing and therefore all other nodes above it must simply be equal to this. □

3.8 Rational Perfect Tree Forcing

The last forcing notion I look at is Miller's rational perfect tree forcing, $\mathbb{PT}$. Recall that $\mathbb{PT}$ is the set of perfect trees $T \subseteq \omega^{<\omega}$ so that for all $s \in T$ there is a $t \supseteq s$ with ω-many immediate successors. The order is inclusion and the unique branch through the trees in the generic is called a *Miller real*. Let us denote such a real by m.

Theorem 3.26. *Let m be a Miller real over W. Then the $\leq_W$ diagram in $W[m]$ is determined by $\emptyset = \mathcal{B}(\neq^*) = \mathcal{D}(\neq^*)$ and all other nodes are equal to the set of all new reals.*

Proof. There are three things to show: that $\mathbb{PT}$ adds no eventually different reals, that $\mathbb{PT}$ adds no infinitely often equal reals, that every subforcing of $\mathbb{PT}$ adds an unbounded real. All of these are standard facts about $\mathbb{PT}$. The fact that $\mathbb{PT}$ adds no eventually different real follows immediately from [2, Theorem 7.3.46, Part 1]. The proof that $\mathbb{PT}$ adds no infinitely often equal real is the same as for Laver forcing as $\mathbb{PT}$ also enjoys the

[3] I would like to thank Professor Martin Goldstern who provided the correct proof of this fact for me on Mathoverflow, https://mathoverflow.net/questions/287977/does-laver-forcing-add-an-infinitely-often-equal-real, after Joel David Hamkins pointed out that my original proof was incorrect.

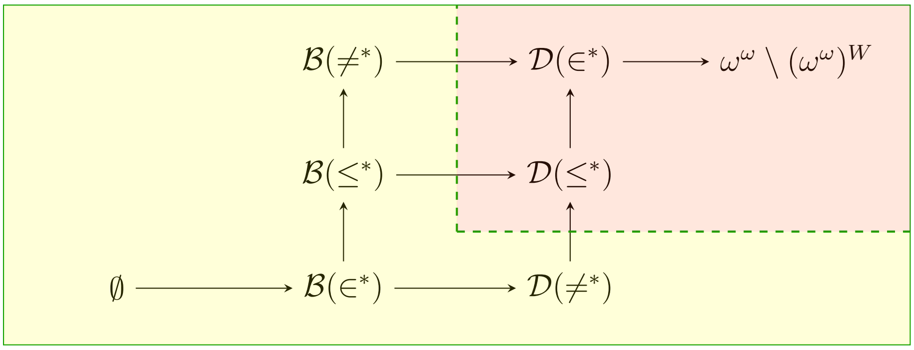


Figure 12: After rational perfect forcing

Laver property ([2, Theorem 7.3.45]). That m is unbounded is clear from the definition of the forcing and so, to finish the theorem it suffices to show that m is of minimal degree. This follows directly from [7, Theorem 3]. □

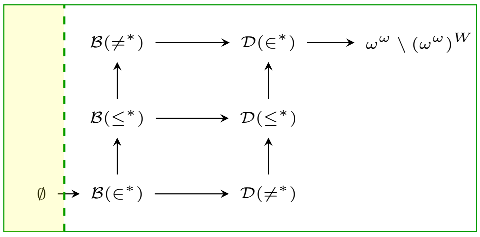
3.9 Cuts in the Diagram and the Analogy with Cardinal Characteristics

Let me finish this section by noting that it follows from what I have shown that the ZFC(W)-provable subset implications implied by Theorem 2.8 are the only ones. In other words, Theorem 3.1 is proved. Indeed a simply inspection of the diagrams above show that every implication shown in Figure 1 is consistently strict and no other implications are true in every V extending W. This shows also that the analogue discussed in the previous section holds in a robust way with the traditional Cichoń diagram. In fact, we can actually show that a stronger fact is true.

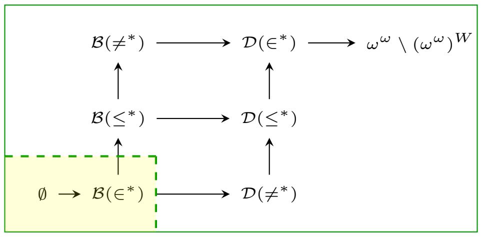
Theorem 3.27. *All cuts consistent with the diagram are consistent with ZFC(W) in the following sense: Given any collection N of (not $\emptyset$)-nodes in the diagram which are closed upwards under $\subseteq$ there is a proper forcing $\mathbb{P}$ in W so that forcing with $\mathbb{P}$ over W results in all and only the nodes in N being nonempty. See Figure 13 for a pictoral representation*

Note that this is slightly weaker than the sense of cuts I have been considering above since I'm making no distinction between various non-empty nodes after forcing.

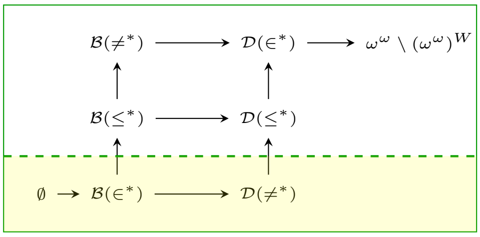
Proof. There are two cuts I have yet to explicitly show. These correspond to e) and i) in Figure 13 below. However for completeness let me go through all cuts one at a time.
a) All nodes are non empty: This is accomplished by $\mathbb{LOC}$.
b) All nodes except $\mathcal{B}(\in^*)$ are non empty: This is accomplished by $\mathbb{D}$.
c) All nodes below $\mathcal{B}(\leq^*)$ are empty and $\mathcal{D}(\neq^*)$ is empty: This is accomplished by $\mathbb{L}$.
d) All nodes below $\mathcal{B}(\neq^*)$ are empty and $\mathcal{D}(\neq^*)$ is non empty: This is accomplished by $\mathbb{E}$.



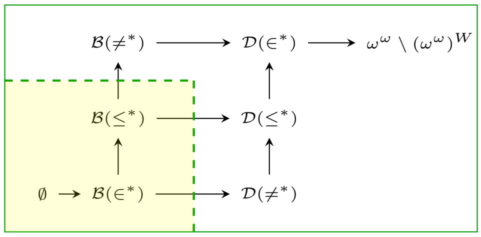
a) All nodes non empty



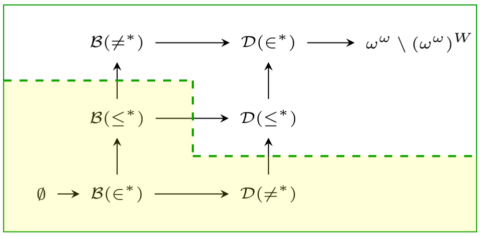
b) All nodes except $\mathcal{B}(\in^*)$ non empty



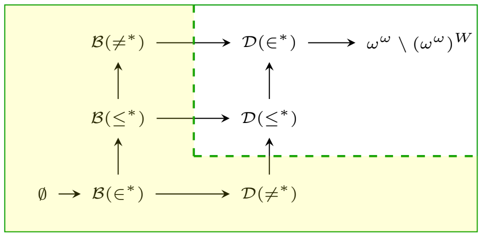
c) All nodes below $\mathcal{B}(\leq^*)$ empty, $\mathcal{D}(\neq^*)$ empty



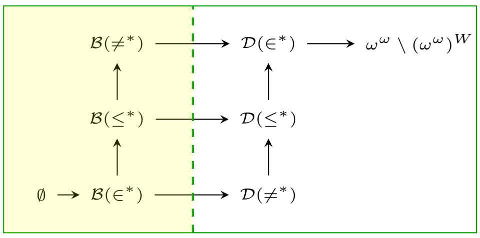
d) All nodes below $\mathcal{B}(\neq^*)$ empty and $\mathcal{D}(\neq^*)$ non empty



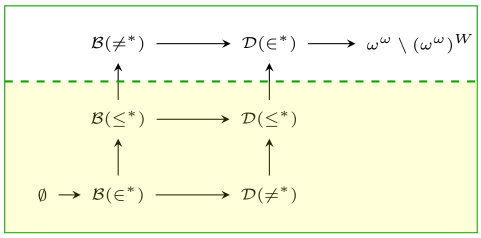
e) All nodes below $\mathcal{D}(\leq^*)$ empty and $\mathcal{B}(\neq^*)$ non empty



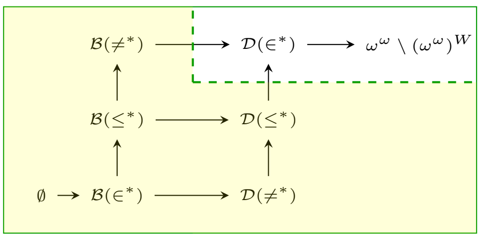
f) All nodes below $\mathcal{D}(\leq^*)$ empty and $\mathcal{B}(\neq^*)$ empty



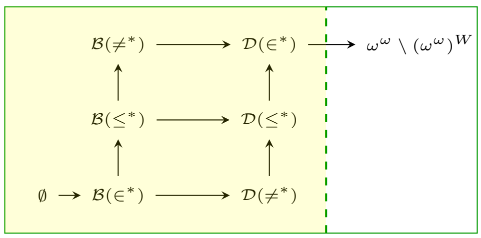
g) All nodes below $\mathcal{D}(\neq^*)$ empty and $\mathcal{B}(\neq^*)$ empty



h) All nodes below $\mathcal{D}(\in^*)$ except $\mathcal{B}(\neq^*)$ are empty



i) All nodes below $\mathcal{D}(\in^*)$ empty



j) All nodes empty except $\omega^\omega \setminus (\omega^\omega)^W$

Figure 13: All Possible Cuts in the $\leq_W$ Cichoń Diagram. Each one can be achieved by a proper forcing over W. White means that the node is not empty while yellow means that it is. No distinction is made between different non-empty nodes. Note that the trivial cut where all nodes remain empty is not shown.

e) All nodes below $\mathcal{D}(\leq^*)$ are empty and $\mathcal{B}(\neq^*)$ is non empty: This is the first case where we still have to prove something. Let $\mathbb{P} = \mathbb{B} * \dot{\mathbb{PT}}$. I claim that in $W^{\mathbb{P}}$ this cut is realized. We have seen that forcing with $\mathbb{B}$ adds an eventually different real and, by further forcing with $\mathbb{PT}$ over $W^{\mathbb{B}}$ will add a real which is unbounded by $W^{\mathbb{B}} \cap \omega^\omega$ and hence $W \cap \omega^\omega$. It remains therefore to see that in $W^{\mathbb{P}}$ there are no dominating or infinitely often equal reals over W. To show that there are no dominating reals, note that in general $\mathbb{PT}$ adds no dominating real, so in $W^{\mathbb{P}}$ there is no real which is dominating over $W^{\mathbb{B}}$. But, since $\mathbb{B}$ is ω^ω-bounding, it follows that there is no real dominating over W in $W^{\mathbb{P}}$. To show there are no infinitely often equal reals, let us first note the following fact.

Fact 3.28 (Corollary 2.5.2 of [2])**.** *Suppose $M \models ZFC$. Then $M \cap 2^\omega \in \mathcal{N}$ if and only if there is a sequence $\langle F_n \subseteq 2^n \mid n < \omega\rangle$ such that $\Sigma_{n=0}^\infty |F_n| 2^{-n} < \infty$ and for every $x \in M \cap 2^\omega$ there are infinitely many n so that $x \restriction n \in F_n$.*

As a corollary of this Fact, notice that adding an infinitely often equal real on ω^ω makes the ground model reals measure 0. To see why, suppose $g \in \omega^\omega$ is infinitely often equal over an inner model M and let $\langle \tau_k \mid k < \omega\rangle$ be an enumeration in M of the elements of $2^{<\omega}$. Then for every $x \in 2^\omega \cap M$ let $\hat{x} : \omega \to \omega$ be defined by $\hat{x}(n) = k$ if $x \restriction n = k$. Clearly if $x \in M$ the $\hat{x} \in M$ so there are infinitely many n such that $\hat{x}(n) = g(n)$. But then, pulling back, let $g' : \omega \to 2^{<\omega}$ be defined by $g'(n) = \sigma_k$ if $g(n) = k$ and $\sigma_k \in 2^n$ and is trivial otherwise. Then we have that for every $x \in M \cap 2^\omega$ if $\hat{x}(n) = g(n)$ then $x \restriction n = g'(n)$ so the sequence $\langle \{g'(n)\} \mid n < \omega\rangle$ witnesses that $2^\omega \cap M$ is measure 0 by the Fact.

From this it follows immediately that $\mathbb{P}$ does not add infinitely often equal reals since both $\mathbb{B}$ ([2, Lemma 6.3.12]) and $\mathbb{PT}$ ([2, Theorem 7.3.47]) preserve outer measure.
f) All nodes below $\mathcal{D}(\leq^*)$ are empty and $\mathcal{B}(\neq^*)$ is empty: This is accomplished by $\mathbb{PT}$.
g) All nodes below $\mathcal{D}(\neq^*)$ are empty and $\mathcal{B}(\neq^*)$ is empty: This is accomplished by $\mathbb{C}$.
h) All nodes below $\mathcal{D}(\in^*)$ except $\mathcal{B}(\neq^*)$ are empty: This is accomplished by $\mathbb{B}$.
i) All nodes below $\mathcal{D}(\in^*)$ are empty: This is the second cut where we still have something to prove. To achieve this one we force with the infinitely often equal forcing $\mathbb{EE}$ as defined in [2, Definition 7.4.11]. This forcing is ω^ω-bounding so it doesn't add reals to $\mathcal{D}(\leq^*)$, does not make the ground model reals meager (both of these facts are proved as part of [2, Lemma 7.4.14]) so it doesn't add reals to $\mathcal{B}(\neq^*)$ and generically adds a real which is infinitely often equal to all ground model elements of the product space $\Pi_{n<\omega} 2^n$. Let's see that $\mathbb{EE}$ adds a real to $\mathcal{D}(\in^*)$. Recall that this means there is a real which is not eventually captured by any ground model slalom. Let $g : \omega \to 2^{<\omega}$ be the infinitely often equal real added by the generic and fix an enumeration $\langle \tau_n \mid n < \omega\rangle$ (in W) of $2^{<\omega}$. Let $\hat{g} : \omega \to \omega$ be the function defined by $\hat{g}(n) = k$ if $g(n) = \tau_k$. I claim that this $\hat{g}$ is as needed. To see why, let $\sigma \in W$ be a slalom. We can associate (in W) a function $f_\sigma : \omega \to 2^{<\omega}$ by letting $f_\sigma(n)$ be τ_k where k is the least so that $k \notin \sigma(n)$ and $\tau_k \in 2^n$. Note that such a k exists since $|\sigma(n)| = n$. Since $f_\sigma \in W$ there are infinitely many n so that $f_\sigma(n) = g(n)$. Therefore there are infinitely many n so that $\hat{g}(n) \notin \sigma(n)$, as needed.
j) All nodes except $\omega^\omega \setminus (\omega^\omega)^W$ are empty: This is accomplished by $\mathbb{S}$.

k) All nodes are empty: This one is not pictured in Figure 13 since it is trivial. Let $\mathbb{P}$ be any forcing not adding reals, such as trivial forcing.

$\square$

To finish this section, let me observe one more analogue with the standard Cichoń diagram. Traditionally in the study of cardinal invariants of the continuum one sandwiches the nodes in Cichoń's diagram on one side by $\aleph_1$, the smallest possible value of any node, and on the other side by $2^{\aleph_0}$, the largest possible value of any node. One then views, for a given model M of ZFC, the values of the other nodes on the diagram for M as a measure of how much these two cardinals vary in M with regards to substantive, mathematical applications. My diagram also natually sandwiches itself between two invariants: the empty set, the smallest possible value of any node, and the entirety of the new reals, $\omega^\omega \setminus (\omega^\omega)^W$, the largest possible value of any node. As such, I view my diagram studied in this paper as measuring, similar to the case of the Cichoń diagram, the difference between the reals of the inner model W and the reals of V. A natural question to ask, therefore, is how strong this "measurement" analogy is between these two diagrams. For example, in the generic extension of W by more than $\aleph_1$ many Cohen reals, all nodes on the right side of the Cichoń diagram equal to $2^{\aleph_0}$ and all nodes on the left equal to $\aleph_1$, paralleling the situation I described for the model $W[c]$. However, in similar models studied for Hechler and eventually different forcing, the nodes in the Cichoń diagram still split into two cardinals, $\aleph_1$ and $\aleph_2$, whereas the diagram discussed in this paper automatically splits in three different sets of reals, as discussed. It appears that this may be necessary due to a result of Khomskii and Laguzzi in [9] stating that there is a cannonical forcing in a certain sense to add infinitely-often-equal reals and this forcing does not add dominating reals, suggesting that perhaps there is no way that both $\mathcal{B}(\leq^*)$ and $\mathcal{D}(\neq^*)$ can be nonempty and equal.

4 Achieving a Full Separation in the $\leq_W$-Cichoń Diagram and the axiom $CD(\leq_W)$

In this section building off the work done in the last section I build a model where there is complete separation between all elements in the diagram.

Theorem 4.1. *(GBC) Given any transitive inner model W of ZFC, there is a proper forcing notion $\mathbb{P}$, such that in $W^{\mathbb{P}}$ all the nodes in the $\leq_W$-Cichoń diagram are distinct and every possibile separation is simultaneously realized.*

In what follows I call the axiom "All consistent separations of the $\leq_W$-diagram are distinct" $CD(\leq_W)$ or "full Cichoń Diagram for $\leq_W$". Thus the above theorem states that $CD(\leq_W)$ can be forced over W by a proper forcing. For different inner models W the sentence $CD(\leq_W)$ may vary but they can all be forced the same way.

Before proving this theorem I need a simple technical result about Sacks and Laver forcing.

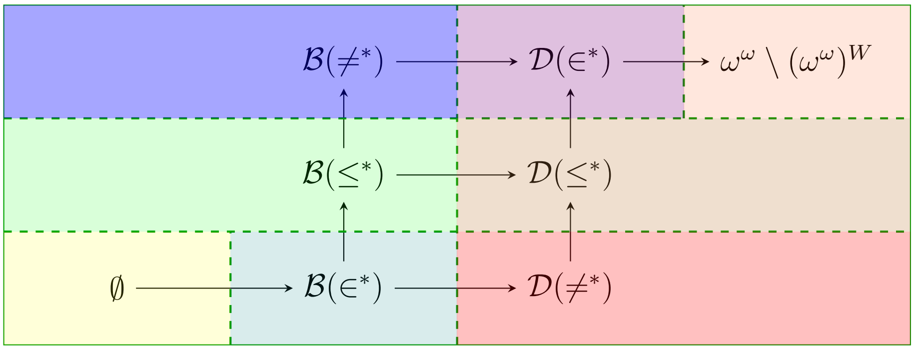


Figure 14: Full Separation of the $\leq_W$-diagram

Lemma 4.2. *The product forcing $\mathbb{S} \times \mathbb{L}$ satisfies Axiom A and hence is proper.*

Proof. Theorem 1 of [7] gives a general framework for showing that certain arboreal forcings satisfy Axiom A (including Sacks and Laver forcings) and here I adapt the proof to the case of a product of two arboreal forcings. Recall that if $p, q \in \mathbb{S}$ and $n \in \omega$ then we let $q \leq_n^{\mathbb{S}} p$ if and only if $q \subseteq p$ and every n^{th} splitting node of q is an n^{th} splitting node of p i.e. if $\tau \in q$ is a splitting node with n splitting predecessors in q then the same is true of τ in p. Also, given a canonical enumeration of $\omega^{<\omega}$ in which σ appears before τ if $\sigma \subseteq \tau$ and $\sigma^\frown k$ appears before $\sigma^\frown(k+1)$ then for $p \in \mathbb{L}$ one gets an enumeration of the elements of p above the stem, $\sigma_1^p, ..., \sigma_k^p, ...$ and if $p, q \in \mathbb{L}$ and $n \in \omega$ then let $q \leq_n^{\mathbb{L}} p$ if and only if $q \subseteq p$ and $s_i^p = s_i^q$ for all $i = 0, ..., n$. Clearly if for every $n \in \omega$ and $(p_s, p_l), (q_s, q_l) \in \mathbb{S} \times \mathbb{L}$ we let $(q_s, q_l) \leq_n (p_s, p_l)$ if and only if $q_s \leq_n^{\mathbb{S}} p_s$ and $q_l \leq_n^{\mathbb{L}} s_l$ then this satisfies the first requirement of Axiom A forcings. Thus, it remains to show that for every $\mathbb{S} \times \mathbb{L}$-name $\dot{a}$ and condition $(p_s, p_l) \in \mathbb{S} \times \mathbb{L}$ if $(p_s, p_l) \Vdash \dot{a} \in \check{V}$ then for every n there is a (q_s, q_l) and a countable set $A \in V$ such that $(q_s, q_l) \Vdash \dot{a} \in A$.

Fix such a name $\dot{a}$ and condition $p = (p_s, p_l)$. Let $D \subseteq \mathbb{S} \times \mathbb{L}$ be the set of all $(q_s, q_l) \leq p$ such that there is some $a(q) \in V$ with $(q_s, q_l) \Vdash \check{a(q)} = \dot{a}$. This set is dense below p since p forces $\dot{a}$ to be an element of V. Let $H_D \subseteq p$ be the set of all pairs $(\sigma, \tau) \in p$ such that there is a $(\sigma', \tau') \subseteq (\sigma, \tau)$ with σ' n-splitting in p_s and τ' n-splitting in p_l and there is some $r_{\sigma,\tau} = (r_s, r_l) \leq p$ in D whose stem (i.e. the pair of the stems from the two components) is (σ, τ). Finally let $Min(H_D)$ be the set of $(\sigma, \tau) \in H_D$ which are minimal with respect to inclusion. Note that $Min(H_D)$ is an antichain since no two elements can be comparable and both minimal. Let $r = (r_s, r_l) = \bigcup\{r_{\sigma,\tau} \mid (\sigma, \tau) \in Min(H_D)\}$. A routine check shows that the set r is a condition in $\mathbb{S} \times \mathbb{L}$ and $r \leq_n p$.

Now let $A = \{a(r_{\tau,\sigma}) \mid (\sigma, \tau) \in Min(H_D)\}$. This set is countable thus to finish the lemma it suffices to show that $r \Vdash \dot{a} \in \check{A}$. To see this, suppose that $t \leq r$ and $t \Vdash \dot{a} = \check{a}$ for some a. By extending t if necessary one may assume that the stem of t is in H_D. But then some initial segment of the stem is in $Min(H_D)$ so $a \in A$, as needed. $\square$

Now I prove Theorem 4.1.

Proof of Theorem 4.1. This essentially follows from the theorems of the previous section.

Given a definable forcing notion $\mathbb{Q}$ let me write $\mathbb{Q}^W$ for the version of that forcing notion as computed in W. Let $\mathbb{P} = \mathbb{S}^W \times \mathbb{L}^W \times \mathbb{LOC}^W \times \mathbb{B}^W$. Then in $W^{\mathbb{P}}$ not every new real is in an element of the diagram since Sacks reals were added. Moreover, by our arguments above the combination of $\mathbb{B}$, $\mathbb{LOC}$ and $\mathbb{L}$ will add reals to every node of the diagram but, none of them will be equal and moreover every possible non-separation is realized as one observes by my previous arguments.

If all I wanted to do was prove the consistency of $CD(\leq_W)$ then I would be done but I also want to show that this forcing is proper and hence ω_1 is preserved. To do this let me first notice that $\mathbb{P}$ can equally be written in terms of iterations of check names of the forcing notions as computed in W i.e. in W one can write $\mathbb{P} = \mathbb{S} \times \mathbb{L} * \check{\mathbb{LOC}}^W * \check{\mathbb{B}}^W$. Since each of these forcings are proper in the ground model I simply need to show that this properness is not killed by the previous forcings in the iteration and therefore $\mathbb{P}$ is the finite iteration of proper forcings and hence proper. I take these one at a time. First by Lemma 4.2 note that $\mathbb{S} \times \mathbb{L}$ is proper. For $\mathbb{B}$ and $\mathbb{LOC}$ recall that these forcings are σ-linked and hence indestructibly ccc. Therefore $\mathbb{P}$ is proper, as desired. □

Let me finish this paper by briefly studying the axiom $CD(\leq_W)$. First, let me show that there are other ways to obtain it. Indeed there is another, less finegrained approach to forcing $CD(\leq_W)$. To describe this, let me make the following simple observation. Recall that the *Maximality Principle* MP of [8] states that any statement which is forceably necessary or can be forced to be true in such as a way that it cannot become later forced to be false, is already true. If Γ is a class of forcings then the maximality principle for Γ, MP_Γ, states the same but only with respect to forcings in Γ.

Proposition 4.3. *The axiom $CD(\leq_W)$ is forceably necessary, that is once it has been forced to be true it will remain so in any further forcing extension. Thus in particular it is implied by the maximality principle, MP and even the maximality principle for proper forcings, MP_{proper}.*

Proof. This is more or less immediate from the definition. Since $CD(\leq_W)$ is defined relative to a fixed inner model and the diagram for W concerns only the models $W[x]$ for $x \in \omega^\omega \cap V$, notice that forcing over V cannot change the theories of the models $W[x]$ for $x \in V$ hence if $CD(\leq_W)$ is true in V it must remain so in any forcing extension. In other words absoluteness for membership in each of the various classes holds and this guarentees that forcing cannot change the relation $x \in A$ for any node A of the diagram.

Since $CD(\leq_W)$ is forceably necessary and indeed can be forced by a proper forcing it follows that MP and MP_{proper} both imply $CD(\leq_W)$. □

Now notice that since all the forcing notions used in Theorem 4.1 have size at most $2^{\aleph_0}$ it follows that the collapse forcing $Coll(\omega, < (2^{2^{\aleph_0}})^+)$ will add a generic making $CD(\leq_W)$ true. Since $CD(\leq_W)$ is forceably necessary it follows that the full collapse forcing cannot kill the generic once it is added and, as a result one obtains

Corollary 4.4. $W^{Coll(\omega, <(2^{2^{\aleph_0}})^+)} \models CD(\leq_W)$

Moreover, note that while the forcing described in Theorem 4.1 was proper and hence preserved ω_1 the collapse forcing used above is not. Therefore the following is immediate.

Corollary 4.5. *The statement "the reals of W are countable" is independent of the theory ZFC(W) + $CD(\leq_W)$.*

Since $CD(\leq_W)$ is forceably necessary and hence cannot be killed once it is forced to be true it follows that any sentence which can be forced to be true from any model must be consistent with $CD(\leq_W)$. Such examples include CH, $2^{\aleph_0} = \kappa$ for any κ of uncountable cofinality, Martin's Axiom and its negation, $\diamondsuit$ and its negation, and a wide variety of forcings associated with the classical Cichoń's diagram. In particular, $CD(\leq_W)$ is independent of any consistent assignment of cardinals to the nodes in the Cichoń diagram (cf [2] for a variety of examples of such).

Let me finish now by showing the consistency of a strong version of $CD(\leq_W)$, which was suggested to me by Gunter Fuchs. The idea is to iteratively force with the forcing $\mathbb{P}$ of Theorem 4.1 for long enough that a large collection of inner models W simultaneously satisfy $CD(\leq_W)$.

Theorem 4.6. *Assume $V = L$. Then there is an $\aleph_2$-c.c. proper forcing extension where $2^{\aleph_0} = \aleph_2$ and for every $\aleph_1$-sized set of reals A there is a set of reals $B \supseteq A$ of size $\aleph_1$ so that $CD(\leq_W)$ holds for $W = L[B]$.*

Proof. Assume $V = L$ and let $\vec{\mathbb{P}} = \langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \mid \alpha < \omega_2 \rangle$ be an ω_2-length countable support iteration of copies of the forcing $\mathbb{P}$ from Theorem 4.1 (i.e. $\dot{\mathbb{Q}}_{\alpha+1}$ evaluates to $(\mathbb{P})^{L^{\mathbb{P}_\alpha}}$). Clearly $\vec{\mathbb{P}}$ is proper. Moreover, since CH holds in the ground model and the forcing $\mathbb{P}$ is easily seen to be of size continuum, and does not kill CH it follows that $\vec{\mathbb{P}}$ has the $\aleph_2$-c.c. and every intermediate stage in the iteration preserves CH: $L^{\mathbb{P}_\alpha} \models$ CH for all $\alpha < \omega_2$. However, since reals are added at every stage the final model satisfies $2^{\aleph_0} = \aleph_2$.

It remains to show that for every $\aleph_1$-sized set of reals A there is a set of reals $B \supseteq A$ of size $\aleph_1$ so that $CD(\leq_W)$ holds for $W = L[B]$. Let A be a set of reals of size $\aleph_1$. Then, there is some α so that $A \in L[G_\alpha]$ for G_α be $\mathbb{P}_\alpha$-generic. Note that we can code G_α by a set of reals of size at most $\aleph_1$, say B, and without loss we can assume that $A \subseteq B$ for $L[G_\alpha] = L[B]$. Then at stage $\mathbb{P}_{\alpha+1}$ we added a generic witnessing that $CD(\leq_{L[B]})$ holds. Moreover, by the fact that this statement is forceably necessary, it cannot be killed by the tail end of the iteration so it holds in the final model. □

While it is not entirely clear what consequences we can expect from $CD(\leq_W)$ for an arbitrary W, the stronger version obtained in Theorem 4.6 has several low hanging fruits in this regard. Let me pluck a particularly simple one connecting the constructibility diagram to the standard Cichoń diagram.

Lemma 4.7. *Assume for every $\aleph_1$-sized set of reals A there is a set of reals $B \supseteq A$ of size $\aleph_1$ so that $CD(\leq_W)$ holds for $W = L[B]$. Then all the cardinals in the Cichoń diagram have size at least $\aleph_2$.*

Proof. It suffices to show that $\text{add}(\mathcal{N}) \geq \aleph_2$. Towards this goal, recall Bartoszyński's characterization of $\text{add}(\mathcal{N})$ as the least cardinal κ so that there is a set of reals X of size κ so that no single slalom can capture all the reals in X ([3, Theorem 5.14]). The result is then immediate for, given any set of reals A of size $\aleph_1$, we can find a set $B \supseteq A$ of size $\aleph_1$ and a slalom σ eventually capturing all reals in $L[B]$ by $CD(\leq_{L[B]})$ so $\text{add}(\mathcal{N}) > \aleph_1$. □

5 Open Questions

I finish by collecting the open questions that have appeared throughout this paper. First I ask about the Cichoń diagram for other reduction concepts. Recall that in the case of $\leq_T$, the sets $\mathcal{B}(\in^*)$ and $\mathcal{B}(\leq^*)$ were equal.

Question 1. What strength is required from a reduction $(0, \sqsubseteq)$ on the reals so that $\mathcal{B}_\sqsubseteq(\in^*) \subsetneq \mathcal{B}_\sqsubseteq(\leq^*)$? In particular, can one achieve this with $\leq_A$?

Next I ask about the ZFC(W)-provable relations between the nodes of the $\leq_W$-Cichoń diagram. While I have shown that there are no other implications it is entirely possible that there are other relations more generally.

Question 2. What other ZFC(W)-provable relations are there between the sets in?

My next collection of questions concerns the subforcings of $\mathbb{LOC}$, a topic that deserves more study.

Question 3. What is the forcing adding the eventually different real described in Lemma 3.19? Does it add a dominating real? Note that it must be ccc, in fact σ-linked and add eventually different reals which are bounded by nearly all ground model reals.

Similarly, one might ask whether there is a similarly exotic subforcing of $\mathbb{LOC}$ for adding a dominating real.

Question 4. Does every subforcing of $\mathbb{LOC}$ adding a dominating real add a $\mathbb{D}$-generic real?

Question 5. Does every subforcing of $\mathbb{LOC}$ add a Cohen real?

Finally I conclude with some questions about the axiom $CD(\leq_W)$.

Question 6. What statements are implied by $CD(\leq_W)$? In particular, does it imply that there are W-generics for the forcings to add reals we have discussed (Cohen, random, etc)?

Question 7. How does $CD(\leq_W)$ relate to standard forcing axioms? In particular does $\text{MA}_{\aleph_1}$ imply $CD(\leq_{L[A]})$ for all $\aleph_1$-sized sets of reals A? Does BPFA?

References

[1] Tomek Bartoszyński. Combinatorial aspects of measure and category. *Fundamenta Mathematicae*, 127(3):225–239, 1987.

[2] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line*. A.K. Peters, Wellsley, MA, 1995.

[3] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*, pages 395–489. Springer, Dordrect, 2010.

[4] Jörg Brendle, Andrew Brooke-Taylor, Keng Meng Ng, and André Nies. An analogy between cardinal characteristics and highness properties of oracles. In Xishun Zhao, Qi Feng, Byunghan Kim, and Liang Yu, editors, *Procedings of the 13th Asain Logic Conference*, pages 1–29. World Scientific, Singapore, 2015.

[5] Jörg Brendle and Benedikt Löwe. Eventually different functions and inaccessible cardinals. *Journal of the Mathematical Society of Japan*, 63(1):137–151, 2011.

[6] Moti Gitik and Saharon Shelah. More on simple forcing notions and forcings with ideals. *Annals of Pure and Applied Logic*, 59:219–238, 1993.

[7] Marcia J. Groszek. Combinatorics on ideals and forcing with trees. *The Journal of Symbolic Logic*, 52(3):582–593, September 1987.

[8] Joel David Hamkins. A simple maximality principle. *The Journal of Symbolic Logic*, 68(2):527–550, 2003.

[9] Yurii Khomskii and Giorgio Laguzzi. Full-splitting miller trees and infinitely often equal reals. *Annals of Pure and Applied Logic*, 168:1491–1506, 2017.

[10] Justin Thomas Palumbo. Hechler forcing and its relatives. *PhD Thesis, UCLA*, 2013.

[11] Corey Bacal Switzer. Invariants of measure and category as degrees of constructibility. In Preparation, 2018.